A simple graph construction of semilinear reachability sets of vector addition systems

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A simple graph construction of semilinear reachability sets of vector addition systems

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Abstract: We associate a graph to a vector addition system, whose path-language is the reachability set , when its construction halts. In the semilinear case, its construction halts, and so gives a precise description of the reachability set, instead of the usual Karp & Miller cover. As a corollary, we present a semidecision procedure of semilinearity of reachability sets for vector addition systems.

Un graphe simple pour représenter les ensembles d'états accessibles semi-linéaires des systèmes d'addition de vecteurs.

Résumé: A tout système d'addition de vecteurs nous associons un graphe, dont le langage des chemins est, quand sa construction s'arrête, l'ensemble des états accessibles. Dans le cadre semi-linéaire, cette construction s'arrête effectivement, et ainsi fournit une description précise de l'ensemble des états accessibles, remplaçant la couverture usuelle de Karp & Miller. En corollaire, nous présentons une procédure de semi-décision de la semi-linéarité des ensembles d'états accessibles pour les systèmes d'addition de vecteurs.

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1 Introduction

Vector addition systems were invented by Rabin in order to investigate several questions about Petri nets which were posed to him by Karp & Miller [8]. These last ones conjectured that reachability sets of Petri nets were always semilinear sets which were well known for the study of context-free languages by Ginsburg, for example [5]. Rabin destroyed this conjecture by showing that semilinear sets are a proper subset of vector addition systems, and proving that the inclusion problem of reachability sets is undecidable [1].

Several authors have isolated subclasses of vector addition systems which have semilinear reachability sets in order to solve, in these particular cases, the decision problems of reachability, inclusion or equivalence [6,10,12,13].

We propose a new graph, that we call complete factorization graph, to represent the reachability sets in the semilinear cases. This graph construction is very simple and is directly derivated from the vectors of the systems. We first present some basic definitions and the complete factorization graph itself. Then we show that, if its construction halts, the path-language of the complete factorization graph is the reachability set. At last, we prove that in the case of a semilinear reachability set, the construction halts. As a corollary, we get a semidecision procedure of the semilinearity of the reachability set for vector addition systems.

2 Semilinear sets

We recall here some basic definitions and results of formal language theory and some usual notations charasteristic of commutative monoids [5].

Definition : A monoid consists of a set M, a binary operation on M, usually denoted by multiplication , and which is postulated to be associative, and a unit in M, usually noted 1.

Definition : A monoid M is said to be commutative if for all elements m,n in M, the equality mn=nm holds. In this case, it is common to note the monoid operation additively and to denote the unit by 0.

Definition : A subset A of M is a submonoid of M if $1 \in A$ and $A^2 \subset A$. Given any subset A of M, the set $A^* = \bigcup_{n>0} A^n$, where $A^0 = \{1\}$ and $A^{n+1} = A^n A$, is a submonoid of M. Actually A^* is the least submonoid for the order of set inclusion containing A.

Definition : Let M be a monoid. The family Rat(M) of rational subsets of M is the least family $\mathfrak{R}$ of subsets of M satisfying the following conditions:

(i) $\varnothing \in \mathfrak{R}$; $\{m\} \in \mathfrak{R}$ for all $m \in M$;

(ii) if $A,B \in \mathfrak{R}$, then $A \cup B$, $AB \in \mathfrak{R}$;

(iii) if $A \in \mathfrak{R}$, then $A^* \in \mathfrak{R}$.

We need the commutative monoid $\mathbb{Z}^k$ of k-uples of integers, with the usual addition of vectors (componentwise); its unit 0^k is the element in $\mathbb{Z}^k$, each of whose coordinates is 0. We especially use the commutative monoid $\mathbb{N}^k$ of k-uples of nonnegative integers, with the addition as in $\mathbb{Z}^k$. Its generators are called unit vectors and are all vectors equal to 1 in one coordinate and 0 in all others. We now recall some properties of rational sets in commutative monoids. Remind that the monoid operation is noted +.

Definition : Let M be a commutative monoid. A linear set of M is a set under the form $u+V^*$ where u is in M and V is a finite subset of M. In other words, if $V=\{v_1, ..., v_p\}$, a linear set of M has a representation $\{u + \sum_{i=1}^{p} n_i v_i \,/\, n_i \in \mathbb{N}\}$.

A subset of M is said to be semilinear if it is a finite union of linear sets, so has a representation $\bigcup_{i=1}^{n} (u_i + V_i^*)$.

From the definition of rational subsets of a monoid follows this important result:

Proposition :

Let M be a commutative monoid. A subset of M is rational if and only if it is semilinear.

We now define the notion of residual of a rational subset of a commutative monoid. It is derivated from the usual notion of left quotient.

Definition :

Let M be a commutative monoid over an alphabet X. The residual $a^{-1}.L$ of a rational set L by an element a of X is recursively defined as follows:

- $a^{-1}.\{0\} = a^{-1}.\varnothing = \varnothing$;
- $a^{-1}.\{a\} = \{0\}$; $\forall b \in X, b \neq a,\ a^{-1}.\{b\} = \varnothing$;

If L_1, L_2 are subsets of M:

- $a^{-1}.(L_1 \cup L_2) = a^{-1}.L_1 \cup a^{-1}.L_2$;
- $a^{-1}.(L_1 + L_2) = (a^{-1}.L_1 + L_2) \cup (a^{-1}.L_2 + L_1)$
- $a^{-1}.(L_1^*) = a^{-1}.L_1 + L_1^*$

The extension of the former definition to the residual of a subset L by a word of M is defined as follows:

. $0^{-1}.L = L$. $\forall w \in X^*, \forall a \in X \quad (w+a)^{-1}.L = a^{-1}.(w^{-1}.L)$

More intuitively, the residual of a subset L of M by an element a of M is the set $\{w \,/\, a+w \in L\}$.

3 Vector addition systems

Vector addition systems have been introduced as a mathematical structure to express solutions of some decision problems for Petri nets or parallel schemata [8]. The model is very simple and consists in one only definition.

Definition : A k-dimensional vector addition system is a pair $\mathfrak{A} = (u,A)$ in which u is a k-dimensional vector of nonnegative integers, and A is a finite set of k-dimensional integer vectors. The reachability set, that we will later note $\overset{*}{\underset{R_{\mathfrak{A}}}{\rightarrow}}$, is the set of all vectors of the form $u+a_1+a_2+...+a_p$ such that :

$$a_i \in A \quad i = 1, 2, ..., p \quad \text{and} \quad u+a_1+a_2+...+a_i \geq 0^k \quad i = 1, 2, ..., p.$$

Thus a point is in $\overset{*}{\underset{R_{\mathfrak{A}}}{\rightarrow}}$ if it can be reached from u by a sequence of moves in the set A in such a way that each intermediate point is in the first orthant of k-space, that is to say $\mathbb{N}^k$.

Definition : Given $\mathfrak{A} = (u,A)$ a k-dimensional vector addition system. The test part of a vector a_i of A is the absolute value of its projection on $(-\mathbb{N}^k)$ and is denoted by t_i. The set part of a vector a_i of A is its projection on $\mathbb{N}^k$ and is denoted by s_i.

4 Complete factorization graph

We define the type of graphs we use and the factorization and completion operations on these graphs. These notions are partly inspired by D.Caucal's study on suffix rewritings of words [3].Then we associate graphs and vector addition systems.

Let R be a finite binary relation over $\mathbb{N}^k$. The domain of R, denoted by Dom(R), is the set $\{x \,/\, \exists y \in \mathbb{N}^k \ (x,y) \in R\}$. S is the set $\mathrm{Dom}(R) \cup \{0^k\}$.

We denote $\underset{R}{\rightarrow} = \{(u+x,u+y) \,/ (x,y) \in R\}$, a rewriting step , and $\overset{*}{\underset{R}{\rightarrow}} = (\underset{R}{\rightarrow})^*$ the rewriting relation.

We introduce $\mathbb{G}$, the monoid ($P(S \times P(\mathbb{N}^k) \times S)$, $\circ$) of graphs on S with edges labelled in $P(\mathbb{N}^k)$, with the following composition product $\circ$:

$$A \circ B = \{ (s, L+M, t) \,/\, \exists r : (s, L, r) \in A \text{ and } (r, M, t) \in B \}$$

Its unit is $\{ (s, \{0^k\}, s) / s \in S \}$. We define $A^* = \bigcup_{i=0}^{\infty} A^i$, the graph on S with edges labelled by labels of the finite paths of A. We now call graph an element of $\mathbb{G}$.

Definition : For each finite graph A, the factorization $<A>$ of A is defined as follows:

$$<A> = \{ (s, D, v) \,/\, D = v^{-1}.L_A(s),\ v \in Dom(R) \text{ and } \forall\ (s, D', v) \in A^*:\ \neg(D \sqsubseteq D') \}$$

where for each $s \in S$, $L_A(s)$ is the union of labels of paths from s to 0^k. Definition of residual assures D to be an element of $\mathbb{N}^k$.

Definition : Let $[A]_n$ be the sequence defined by $[A]_0 = A$ et $[A]_{i+1} = [A]_i \uplus <[A]_i>$, where the special union $\uplus$ of two graphs A and B on S with labels in $P(\mathbb{N}^k)$ is defined as follows:

$$A \uplus B = \{(s, X, t) \ / \ X = \cup\{Y / (s, Y, t) \in A \text{ or } (s, Y, t) \in B\} \}$$

The completion $\overline{A}$ of A is the union, for all i, of $[A]_i$.

We define the complete factorization graph G(R) of a relation R in completing a basic graph. This one is the union of two graphs: the first one H(R), is built from pairs of R, the second one I(R) is obtained by factorizing the words of Dom(R).

Definition : Given R a finite relation over $\mathbb{N}^k$, the complete factorization graph G(R) of R is: $\quad G(R) = \overline{H(R) \uplus I(R)}$

where $H(R) = \{(x, T, z) / x, z \in S, T = \cup\{ z^{-1}.\{y\} / (x,y) \in R, x \neq y, y \geq z \} \}$

and $\quad I(R) = \{(x, \{ y^{-1}.\{x\}\}, y) / x, y \in S \}$

We now state that if a graph, including I(R) for a relation R, is complete then any path in the graph can be factorized by any word of S. The proof is easy and is left to the reader.

Lemma : Given R a finite relation over $\mathbb{N}^k$ and A a graph including I(R).

$$<A> = \emptyset \Leftrightarrow \forall (x,U,y) \in A^*, \forall V \in P(\mathbb{N}^k), \forall z \in S \text{ such that } V+z = U+y \ : (x,V,z) \in A^*$$

5 Path-language of the complete factorization graph

We now consider complete factorization graphs linked with a vector addition systems. From a vector addition system $\mathfrak{A} = (u,A)$, we extract the binary relation $R_{\mathfrak{A}}$ between test and set parts of vectors of A : $R_{\mathfrak{A}} = \{ (t_i , s_i) / a_i \in A \}$.
The reachability set of $\mathfrak{A}$ from the initial state u is equal to the labels of the paths $u \to 0^k$ in the path-language of the complete factorization graph of $(R_{\mathfrak{A}} \cup \{(u,u)\})$.

<u>Proposition</u> :
Given a vector addition system $\mathfrak{A}=(u,A)$ and $R_{\mathfrak{A}}$ the finite binary relation over $\mathbb{N}^k$ associated to it . If $G(R_{\mathfrak{A}})$ is computable, that is to say there exists an integer i such that

$<[G(R_{\mathfrak{A}})]_i> = \varnothing$ then: $\underset{R_{\mathfrak{A}}}{\overset{*}{\to}}(u) = \{ v / (u, v, 0^k) \in G(R_{\mathfrak{A}} \cup \{(u, u)\})^* \}$.

<u>proof</u> :

It suffices to show : $\forall u \in \mathrm{Dom}(R_{\mathfrak{A}})$ $\underset{R_{\mathfrak{A}}}{\overset{*}{\to}}(u) = \{ v / (u, v, 0^k) \in G(R_{\mathfrak{A}})^* \}$

$\subseteq$: Let us prove by induction on $n \geq 0$ and for $u \in \mathrm{Dom}(R_{\mathfrak{A}})$, the following inclusion :

$$(\underset{R_{\mathfrak{A}}}{\to})^{(n)}(u) \subseteq \{ v / (u, v, 0^k) \in G(R_{\mathfrak{A}})^* \}$$

<u>Basis</u>: $(\underset{R_{\mathfrak{A}}}{\to})^{(0)}(u) = \{u\}$ and $(u, u, 0^k) \in I(R) \subseteq G(R_{\mathfrak{A}})^*$

<u>Induction step</u>: We have $u \underset{R_{\mathfrak{A}}}{\to}^{(n)} x \underset{R_{\mathfrak{A}}}{\to} y$. By induction hypothesis: $(u, x, 0^k) \in G(R_{\mathfrak{A}})^*$.

By definition of $\underset{R}{\to}$, $x = w+s$, $y = w+t$ and $(s,t) \in R_{\mathfrak{A}}$ hold. We distinguish two cases:

(i) $s = t$: Then $x = y$ and $(u, y, 0^k) \in G(R_{\mathfrak{A}})^*$.

(ii) $s \neq t$: As $(u, x, 0^k) \in G(R_{\mathfrak{A}})^*$ and $x = w+s$, and as it exists i such that $G(R_{\mathfrak{A}}) = [G(R_{\mathfrak{A}})]_i$ with $<[G(R_{\mathfrak{A}})]_i> = \varnothing$, by the former lemma : $(u, w, s) \in G(R_{\mathfrak{A}})^*$.
Furthermore $(s,t) \in R_{\mathfrak{A}}$ so, setting $t= t_1+t_2$: $(s, t_1, t_2) \in H(R_{\mathfrak{A}}) \subseteq G(R_{\mathfrak{A}})^*$.
If $t_2 = 0$ then $(u,y,0^k) = (u,w+t_1,0^k) \in G(R_{\mathfrak{A}})^*$, else by construction $(t_2,t_2,0) \in I(R_{\mathfrak{A}})$ and $I(R_{\mathfrak{A}}) \subseteq G(R_{\mathfrak{A}})$, therefore $(u, y, 0^k) = (u, w+t_1+t_2, 0^k) \in G(R_{\mathfrak{A}})^*$.

$\supseteq$: Consider the set E of graphs such that :

$$A \in E \Leftrightarrow \forall (x, u, y) \in A \; : \; x \xrightarrow[R_{\mathfrak{A}}]{*} u+y \; .$$

Given A and B graphs of E. $A \cup B \in E$. For all integer n , $A^n \in E$ (by induction on n), so $A^* \in E$. <A> and $\overline{A}$ are also graphs of E.

By definition, $H(R_{\mathfrak{A}})$ and $I(R_{\mathfrak{A}})$ belong to E and since $G(R_{\mathfrak{A}}) = \overline{H(R_{\mathfrak{A}}) \cup I(R_{\mathfrak{A}})}$, the graph $G(R_{\mathfrak{A}})^*$ is an element of E. From $(u, v, 0^k) \in G(R_{\mathfrak{A}})^*$, we deduce $u \xrightarrow[R_{\mathfrak{A}}]{*} v$.

❑

6 Computability of the complete factorization graph

It remains to prove that the complete factorization graph of a vector addition system with a semilinear reachability set is computable. Up to now, we have failed to state a direct proof, so we present another procedure to produce the reachability set. Grabowski introduced it as a means to solve the decision problem of persistence in vector addition systems [6]. This explains that his algorithm was designed without regard for simplicity of the semilinear representation neither computational complexity.

This algorithm builds a sequence of semilinear sets and, when the reachability set is semilinear, outputs it as a set S_n. We first prove all the path-languages of intermediate graphs, which are built in the complete factorization graph construction, are in S_n. Then we show that each semilinear set, which leads to the construction of S_n, is included in the path-language of one of the intermediate graphs.

Some definitions are required before announcing Grabowski's lemma.

Definition : Given a vector addition system $\mathfrak{A}=(u,A)$, a sequence $(a_1, a_2, ..., a_p)$ of elements of A and x an element of $\mathbb{N}^k$. We say that $(a_1, a_2, ..., a_p)$ is valid at x if and only if, for each $j \leq p$, $x + a_1 + a_2 + ... + a_j \geq 0^k$.

Definition : Given a vector addition system $\mathfrak{A}$ and d an element of $\mathbb{N}^k$. We say that $\mathfrak{A}$ splits at d if and only if $\xrightarrow[R_{\mathfrak{A}}]{*}(d)$ has a representation : $\xrightarrow[R_{\mathfrak{A}}]{*}(d) = \bigcup_{i=1}^{m} d_i + V_i{*}$ such that $d_i + V_i \subseteq \xrightarrow[R]{*}(d_i)$, for all i, where d_i's are elements and the V_i's finite subsets of $\mathbb{N}^k$.

Lemma [6]: Given a vector addition system $\mathfrak{A} = (u, A)$. There exists an algorithm which, when applied to $\mathfrak{A}$, has the following behaviour:

(i) if $\mathfrak{A}$ splits at u then the algorithm halts.

(ii) if the algorithm halts, then it outputs a semilinear set, and this set is $\xrightarrow[R_{\mathfrak{A}}]{*}(u)$.

Proof : The algorithm proceeds as follows: it enumerates $\xrightarrow[R_{\mathfrak{A}}]{*}(u)$, starting with u, by successive application of valid vectors. Suppose that, at a given stage, this process has resulted in a finite subset $\{y_1,...,y_n\}$ of $\xrightarrow[R_{\mathfrak{A}}]{*}(u)$, containing u, and in a binary relation on this set which is a subrelation of the reachability relation. With each y_j, the algorithm associates the set B_j of repeating vectors. This is the set of all differences y_k-y_j for which $y_j \leq y_k$ holds and for which the pair (y_j,y_k) occurs in the binary relation.

Let S_n be the semilinear set: $S_n = \bigcup_{j=1}^{n} y_j + B_j{*}$.

The algorithm checks whether S_n is closed under the operator $\xrightarrow[R_{\mathfrak{A}}]{*}$, i.e. whether for each $y \in S_n$ and each $a_i \in A$, the validity of (a_i) at y implies that $y + a_i \in S_n$.

This means that we are to check the truth of the formula:

$$\bigwedge_{a_i \in A} \forall y\ (y \in S_n, \text{ and } (a_i) \text{ valid at } y \Rightarrow y + a_i \in S_n)$$

Since this formula is expressible in S1S [2,4], it is decidable. If the test is positive, the algorithm halts and outputs S_n. If not, it proceeds to the next stage, that is to say, choose a valid vector and apply it.

We show that the algorithm satisfies (i) and (ii).

(i) : Let $\overset{*}{\underset{R_{\mathfrak{A}}}{\rightarrow}}(u) = \bigcup_{i=1}^{m} u_i + V_i^*$, with $u_i + V_i \subseteq \overset{*}{\underset{R}{\rightarrow}}(u_i)$ for all i.

The algorithm eventually discovers that for all i, u_i is reachable from u, and all vectors in $u_i + V_i$ are reachable from u_i. Assume that at this stage it has constructed $\{y_1,\ldots,y_n\} \subseteq \overset{*}{\underset{R_{\mathfrak{A}}}{\rightarrow}}(u)$ together with the corresponding sets $B_1,\ldots,B_n$ of repeating vectors.

Every u_i occurs in the set $\{y_1,\ldots,y_n\}$. Let u_i equals y_j. Then V_i is contained in B_j. Hence $\overset{*}{\underset{R_{\mathfrak{A}}}{\rightarrow}}(u) \subseteq S_n$. On the other hand, $S_n \subseteq \overset{*}{\underset{R_{\mathfrak{A}}}{\rightarrow}}(u)$. This implies $\overset{*}{\underset{R_{\mathfrak{A}}}{\rightarrow}}(S_n) \subseteq S_n$. At this stage, the algorithm halts.

(ii) : If the algorithm halts, in the last stage, it has computed S_n. It has found that S_n is closed under $\underset{R_{\mathfrak{A}}}{\rightarrow}$. Since $u \in S_n \subseteq \overset{*}{\underset{R_{\mathfrak{A}}}{\rightarrow}}(u)$, this implies $S_n = \overset{*}{\underset{R_{\mathfrak{A}}}{\rightarrow}}(u)$. ❑

We are now able to state the halting property of our complete factorization graph construction:

<u>Proposition</u> :

Given a vector addition system $\mathfrak{A}=(u,A)$ and $R_{\mathfrak{A}}$ the finite binary relation over $\mathbb{N}^k$ associated to it. If the set of all vectors reachable from u is semilinear then the complete factorization graph construction of $(R_{\mathfrak{A}} \cup \{u,u\})$ halts.

<u>proof</u> :

Given the sequence S_m of semilinear sets built by Grabowski's algorithm and S_n, its last element equal to $\overset{*}{\underset{R_{\mathfrak{A}}}{\rightarrow}}(u)$. On the other hand, we define a sequence G_m of graphs as follows: $G_0 = H(R_{\mathfrak{A}}) \uplus I(R_{\mathfrak{A}})$ et $G_{i+1} = G_i \uplus \langle G_i \rangle$.

Let $|G_i^*|$ the sets of labels of the paths $u \rightarrow 0^k$ in G_i^*. We have $G_{i+1} = G_i$ if and only if $|G_i^*| = \overset{*}{\underset{R_{\mathfrak{A}}}{\rightarrow}}(u)$. Define $(x,y,z) \in G_i^*$ if $\exists y' : y \subseteq y'$ and $(x,y',z) \in G_i^*$.

(1) : $\forall i \in \mathbb{N}\ |G_i^*| \subseteq \xrightarrow[R_{\mathfrak{g}}]{*}(u) = S_n$. See ($\supseteq$) of the previous proposition.

(2) : $\forall i \in \mathbb{N}$, $\exists j \in \mathbb{N} : S_i \subseteq |G_j^*|$. By induction on i.

Basis : $S_0 = \{u\}$. $S_0 \subseteq |G_0^*|$ par définition de $I(R_{\mathfrak{g}} \cup \{u,u\})$

Induction step : $\forall h \in \mathbb{N}$, $h \leq i$, $\exists f \in \mathbb{N}$, $S_h \subseteq |G_f^*|$.

By induction hypothesis, it exists f such that $S_i \subseteq |G_f^*|$.

For all x in S_{i+1}, we distinguish three cases:

a) $x \in S_i : (u, x, 0^k) \in G_f^* \subseteq G_{f+1}^*$.

b) $x \in S_{i+1} - S_i$ and $\exists j\ x \in y_j + B_j{}^*$ where $y_j \in S_{i+1}$.

There is a finite subset $\{B_j^1, ..., B_j^r\}$ of vectors of B_j such that :

$$x = y_j + n_1 B_j^1 + ... + n_r B_j^r, \quad n_1, ..., n_r \in \mathbb{N}.$$

(i) We are to show that it exists an intermediate graph G_z containing the path from u to 0^k and labelled by $y_j + B_j^p$ for all p.

For any p in [1,r], there exists a sequence $(a_1, a_2, ..., a_q)$ of elements of A, valid at y_j, since $y_j + B_j^p$ is reachable from y_j by construction of B_j, such that $a_1 + a_2 + ... + a_q = B_j^p$. Let $t_1, t_2, ..., t_q$ (respectively $s_1, s_2, ..., s_q$) be the test (resp. set) parts of $a_1, a_2, ..., a_q$.

Since $(u, y_j, 0^k) \in G_f^*$ and $y_j + a_1 \geq 0^k$ then $(u, t_1^{-1}.y_j, t_1)$ may not belong to G_{f+1}^* but we are allowed to say :

$$(u, t_1^{-1}.y_j, t_1) \in (G_{f+1}^* \cup \langle G_{f+1}^* \rangle)^* = G_{f+2}^*$$

So we have $(u, t_1^{-1}.y_j, t_1) \in G_{f+1}^*$ and $(t_1, s_1, 0^k) \in G_0^*$.

Thus: $\quad (u, t_1^{-1}.y_j + s_1, 0^k) = (u, y_j + a_1, 0^k) \in G_{f+2}^*$

The same applies for a_2 : $(u, y_j + a_1 + a_2, 0^k) \in G_{f+3}^*$

So we easily deduce : $(u, y_j + a_1 + a_2 + ... + a_q , 0^k) \in G^*_{f+q+1}$

Let (Q-1) be the greatest cardinality of the $(a_1, a_2, ..., a_q)$-sequences linked to the B^P_j's. For all p, $(u, y_j + B^P_j, 0^k)$ belongs to G^*_{f+Q}.

(ii) We have to iterate these vectors B^P_j and so be able to recognize the elements $y_j + nB^P_j$, for all p and n. Since $(u, y_j + B^P_j, 0^k)$ belongs to G^*_{f+Q}, and we know that if $y_j + nB^P_j$ is reachable from y_j for all n, then (a_1) is valid at $y_j + B^P_j$, we deduce : $(u, t_1^{-1}.(y_j + B^P_j), t_1) \in (G_{f+Q} \cup \langle G_{f+Q}\rangle)^* = G^*_{f+Q+1}$

(iii) We are now to chain these iterations of B^P_j's and this way be able to recognize every element $x = y_j + n_1B^1_j + ... + n_rB^r_j$, for all r and $n_1, ..., n_r$.

For all s and v in [r], we have: $(u, y_j + B^s_j, 0^k) \in G^*_{f+Q}$

and there exists a sequence $(a_1, a_2, ..., a_h)$ of elements of A, valid at y_j, such that $a_1 + a_2 + ... + a_h = B^v_j$.

Let $t_1, t_2, ..., t_h$ (respectively $s_1, s_2, ..., s_h$) be the test (resp. set) parts of $a_1, a_2, ..., a_h$. Since $y_j + B^s_j + B^v_j$ is reachable from y_j, (a_1) is valid at $y_j + B^s_j$. Thus :

$$(u, t_1^{-1}.(y_j + B^s_j), t_1) \in (G_{f+Q} \cup \langle G_{f+Q}\rangle)^* = G^*_{f+Q+1}$$

And so by combination of iterations and changes of vectors, we obtain:

$$(u, y_j + n_1B^1_j + ... + n_rB^r_j, 0^k) \in G^*_{f+Q+1}$$

<u>To conclude</u>, it does exist an integer z such that $|G^*_z| = S_n = \xrightarrow[R_{\mathfrak{A}}]{*}(u)$. ❑

7 Complexity considerations

The completion of the graph $H(R_{\mathfrak{A}}) \cup I(R_{\mathfrak{A}})$ requires the construction of a sequence of intermediate graphs produced by successive factorizations.

A factorization consists in $|dom(R_{\mathfrak{A}})|$ rational set computations and $|dom(R)|^2$ computations of residuals, to which we are to add $|dom(R_{\mathfrak{A}})|^2$ inclusion tests of two rational sets. This last operation is obtained by an emptyness test of the intersection between the first set and the complement of the other one. This problem is NP-complete [7].

The number of intermediate graphs can grow as much as the number of vertices of the coverability graph [8] in pathologic cases, as seen in [14]: Vector addition systems like those which compute Ackermann functions have very large but finite reachability sets, so there will not be any loop in our graphs. Thus reachable vectors will be added to labels ones after the others, just like in a systematic exploration of the reachability set.

8 A new semidecision procedure of the semilinearity for vector addition systems reachability set

As a corollary of path-language and halting results and a résumé, we present an algorithm to build in a simple manner the reachability set of vector addition systems, in the semilinear cases:

- Given $\mathfrak{A} = (u,A)$ a k-dimensional vector addition system,
- Build the set of test and set parts of vectors of A, and so $R_{\mathfrak{A}}$,
- Compute the complete factorization graph of $R_{\mathfrak{A}} \cup \{(u,u)\}$,
- If the construction halts, then the path-language from the u-labelled vertex to the

0^k-labelled one equals $\xrightarrow[R_{\mathfrak{A}}]{*}(u)$.

Without any change, this algorithm, and the complete factorization graph construction itself, suit to other similar models like Petri nets or vector replacement systems [9]. Self-loop transitions, which does not appear explicitly in vector addition systems are expressible as well as other transitions in our binary relation formalism.

9 Further questions

The decision problem of semilinearity for reachability sets remains open. To establish a direct proof of the halting problem may help in this way. Would it be possible to determine a condition of non semilinearity and a bound of the exploration of the reachability set which assures the detection of this property ? Up to now, we failed to answer but we give few hints in [11]. We feel that a positive answer to the semilinearity problem would be of high interest for many applications.

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